Curves 1.1

We say f is *smooth* if f is C^k for every integer k.

A *parameterized* curve is a C^n (or smooth) map $\alpha : I \to \mathbb{R}^n$ for some interval $I = (a, b)$ or $[a, b]$ in $\mathbb{R}$.

We say α is *regular* parameterized curve if it is a parameterized curve and if $\alpha^{'} \neq 0$ for all $t \in I$.

The velocity vector $\alpha'(t)$ is tangent to the curve at $\alpha(t)$ and its length $||\alpha'(t)||$, is the speed of the particle.

The *arclength* of a curve α from a to t is given by $s(t) = \int_a^t ||\alpha'(t)||du$.

The curve α is *arclength parametrized* if $s(t) = t$ for all t. Equivalently, if $||\alpha'(t)|| = 1$.

Examples of curves.

Given points $\vec{P}$ and $\vec{Q}$, the parameterization of a line going from $\vec{P}$ to $\vec{Q}$ is given by: $\alpha(t) = P + t(Q - P)$ where $t \in \mathbb{R}$ and $0 \leq t \leq 1$.

The parameterization of the ellipse is given by $\alpha(t) = (a\cos(t), b\sin(t))$ where $t \in \mathbb{R}$ and $0 \leq t \leq 2\pi$.

The *cuspidal cubic* parameterization is given by $\alpha(t) = (t^2, t^3)$ and the *nodal cubic* is given by $\beta(t) = (t^2 - 1, t^3 - t)$.

The *twisted cubic* (in $\mathbb{R}^3$) is given by $\alpha(t) = (t, t^2, t^3)$ where $t \in \mathbb{R}^3$. It's projections in the $xy-, xz-$ and $yz-$ coordinate planes are given by $y = x^2$, $z = x^3$ and $z^2 = y^3$ (cuspidal cubic).

The *cycloid* is given by $\alpha(t) = a(t - \sin(t), 1 - \cos(t))$ where $t \in \mathbb{R}$.

The *helix* (in $\mathbb{R}^2$) is given by $\alpha(t) = (a\cos(t), a\sin(t), bt)$.

The *catenary* is given by the graph of $f(x) = C\cosh(\frac{x}{C})$, for any constant $C > 0$ or by the parametrization $\alpha(t) = (t, \cosh(t))$.

The *tractrix* is given by $\beta(t) = (t - tanh(t), sech(t))$ for all $t \geq 0$.

Important properties of cosh and sinh.

$$\cosh(t) = \frac{e^t + e^{-t}}{2}, \qquad \sinh(t) = \frac{e^t - e^{-t}}{2}, \qquad tanh(t) = \frac{\sinh(t)}{\cosh(t)}, \qquad sech(t) = \frac{1}{\cosh(t)}$$

$$\cosh^2(t) - \sinh^2(t) = 1, \qquad tanh^2(t) + sech^2(t) = 1$$
$$\sinh'(t) = \cosh(t), \qquad \cosh'(t) = \sinh(t), \qquad tanh'(t) = sech^2(t)$$
$$sech'(t) = -tanh(t) \cdot sech(t).$$

Curves Continued

Let $f : (a, b) \to \mathbb{R}^3$ be differentiable. Then $||f(t)||$ is constant for all t if and only if $f(t) \cdot f'(t) = 0$.

Assume that the curve α is arclength parameterized. Then $\mathbf{T}(t) = \alpha'(t)$ is the *unit tangent vector* to the curve. Note that $\mathbf{T}$ has constant length and $\mathbf{T}'$ is orthogonal to $\mathbf{T}$.

Assume $\mathbf{T}' \neq 0$, then we define the *principal normal vector* $\mathbf{N}(t) = \frac{\mathbf{T}'}{||\mathbf{T}'||}$ and the *curvature* is defined to be $\mathcal{K}(t) = ||\mathbf{T}'(t)||$. Note, that if α is a regular parametrized curve, then $\mathcal{K} = \frac{||\alpha' \times \alpha''||}{||\alpha'||^3}$

Assume $\mathcal{K} \neq 0$. We define *binormal vector* to be $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.

We define the *torsion* to be $\tau(t) = \mathbf{N}' \cdot \mathbf{B}$.

The *Frenet Apparatus* is defined to be the collection $\mathbf{T}, \mathbf{N}, \mathbf{B}, \mathcal{K}$ and τ.
The *Frenet Frame* is given by the formulas:

$$\begin{aligned}
\mathbf{T}'(s) &= \mathcal{K}(s)\mathbf{N}(s) \\
\mathbf{N}'(s) &= -\mathcal{K}(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s) \\
\mathbf{B}'(s) &= -\tau(s)\mathbf{N}(s)
\end{aligned}$$

The *involute* of a curve $\alpha(t)$ is given by $\beta(t) = \alpha(t) + (c - t)\mathbf{T}(t)$ for some constant c. The *evolute* of a curve $\alpha(t)$ is given by $\beta(t) = \alpha(t) + \frac{1}{\mathcal{K}}\mathbf{N}(t) + \frac{1}{\mathcal{K}}\cot(\int \tau dt)\mathbf{B}(t)$.

Assume $\mathcal{K}(t) \neq 0$ for a curve α. Then, its *radius of curvature* is defined as $\rho(s) = \frac{1}{\mathcal{K}(t)}$. Furthermore, we define the *center of curvature* c to be given by the formula: $c = \alpha(t) + \rho(t)\mathbf{N}(t)$ where $\mathbf{N}$ is the normal to the curve.

If a simple closed curve C has length L and encloses area A, then the *isoperimetric inequality* states that $L^2 \geq 4\pi A$ where the equality holds if and only if C is a circle.

Surfaces

Let U be an open set in $\mathbb{R}^2$. A *regular parameterization* of a subset $M \subset \mathbb{R}^3$ is a C^3 injective function $x : U \to M \subset \mathbb{R}^3$ so that $x_u \times x_v \neq 0$. A connected subset $M \subset \mathbb{R}^3$ is called a *surface* if each point has a neighbourhood that is regularly parameterized.

Examples of Surfaces

The graph of a function $f : U \to \mathbb{R}$, $z = f(x, y)$, is parameterized by $x(u, v) = (u, v, f(u, v))$.

The *helicoid* is given by $x(u, v) = (u\cos(v), u\sin(v), bv)$ for $u > 0$ and $v \in \mathbb{R}$. Note that the $u-$curves are called *rays* and the $v-$curves are called *helices*.

The *torus* (i.e. the surface of a doughnut) is given by the regular parametrization $x(u, v) = ((a + b\cos(u))\cos(v), (a + b\cos(u))\sin(v), b\sin(u))$ for $0 \leq u$ and $v < 2\pi$.

The standard parametrization of the unit sphere σ is given by spherical coordinates $(\phi, \theta) \leftrightarrow (u, v)$: $x(u, v) = (\sin(u)\cos(v), \sin(u)\sin(v), \cos(u))$ for $0 < u < \pi$ and $0 \leq v < 2\pi$.

Let $I \subset \mathbb{R}$, and let $\alpha(u) = (0, f(u), g(u)), u \in I$, be a regular parametrized plane curve (injective) with $f > 0$. Then the *surface of revolution* obtained by rotating α about the $z-$axis is parametrized by $x(u, v) = (f(u)\cos(v), f(u)\sin(v), g(u))$ where $u \in I$ and $0 \leq v < 2\pi$. Note, the $u-$curves are called *profile curves* or *meridians*. The $v-$curves are circles, called *parallels*.

Let $I \subset \mathbb{R}$ be an interval, let $\alpha : I \to \mathbb{R}^3$ be a regular parametrized curve, and let $\beta : I \to \mathbb{R}^3$ be an arbitrary smooth function with $\beta(u) \neq 0$ for all $u \in I$. We define a parametrized surface by $x(u, v) = \alpha(u) + v\beta(u)$ for $u \in I$ and $v \in \mathbb{R}$. This is called a *ruled surface* with *rulings* $\beta(u)$ and *directrix* α. The cylinder, helicoid and cone are all examples of a ruled surface.

First Fundamental Form

Let M be a regular parametrized surface, and let $P \in M$. Then choose a regular parametrization $x : U \to M \subset \mathbb{R}^3$ with $P = x(u_0, v_0)$. We define the *tangent plane* of M at P to be the subspace $T_P M$ spanned by x_u and x_v evaluated at (u_0, v_0).

The *unit normal* $\mathbf{n}$ to a parametrized surface is given by $\mathbf{n} = \frac{x_u \times x_v}{||x_u \times x_v||}$. Note $x_u \times x_v$ is a non zero vector orthogonal to the plane spanned by x_u and x_v.

The *first fundamental form* is $I_P(U, V) = U \cdot V$ for $U, V \in T_P M$. To find I_P, we define the equations:

$$
\begin{aligned}
E &= I_P(x_u, x_u) = x_u \cdot x_u \\
F &= I_P(x_u, x_v) = x_u \cdot x_v = x_v \cdot x_u = I_P(x_v, x_u) \\
G &= I_P(x_v, x_u) = x_v \cdot x_v
\end{aligned}
$$

so that I_P is given by the symmetric matrix:

$$
I_P = \begin{pmatrix} E & F \\ F & G \end{pmatrix}
$$

The *surface area* of the parametrized surface $x : U \to M$ is given by the formula: $\int_U ||x_u \times x_v|| dudv = \int_U \sqrt{EG - F^2} dudv$

A map is an *isometry* if it preserves distance and angles. A map is *conformal* if it preserves angles. Note that a map $x(u, v)$ is conformal if and only if $E = G$ and $F = 0$ and $x(u, v)$ is isometric if and only if $E = G = 1$ and $F = 0$.

Second Fundamental Form