MATC34: Complex Numbers Lecture Notes

Author Here

Address Here
E-mail address: aaaaaaaaa@mail.utoronto.ca

Contents

Chapter 1. Power Series 1
1.1. Power Series 1

Chapter 1

Power Series

1.1. Power Series

The power series in $\mathbb{C}$ is as follows:

$$\sum_{n=1}^{\infty} a_n z^n$$

where $z \in \mathbb{C}$, and $a_n \in \mathbb{C}$. This series just like the geometric series, can be explicitly calculated given some certain conditions.

Theorem 1.1. *Given a power series $\sum_{n=1}^{\infty} a_n z^n$, there exists a $0 \leq R \leq \infty$ such that the power series converges for z such that $|z| < R$ and does not converge for $|z| > R$. Moreover, if we let $\frac{1}{0} := \infty$ and $\frac{1}{\infty} := 0$, then R is given by:*

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}}$$

Some terminology before we begin the proof.

Definition 1.2. The disk $\{z : |z| < R\}$ is called the **disk of convergence** for the corresponding power series. Also, R is called the **radius of convergence**.

Proof. Let $L = \frac{1}{R}$. Suppose $|z| < R$. We want to show that $\sum |a_n||z|^n$ converges. Since $|z| < R$, we can choose a $\epsilon > 0$ such that:

$$(L + \epsilon)|z| = r < 1$$

Since $L = \limsup |a_n|^{\frac{1}{n}}$, this means that for sufficiently large n, $|a_n|^{\frac{1}{n}} < (L + \epsilon)$ or equivalently, $|a_n| < (L + \epsilon)^n$. Therefore:

$$\sum_{n=1}^{\infty} |a_n||z|^n \leq \sum_{n=1}^{\infty} (L+\epsilon)^n |z|^n \leq \sum_{n=1}^{\infty} ((L+\epsilon)|z|)^n$$

This is a convergent geometric series so we have that $\sum |a_n||z|^n$ converges.
Now suppose $|z| > R$. By the definition of $\limsup$, for any $\epsilon > 0$ there exists

infinitely many a_n satisfying $|a_n|^{\frac{1}{n}} > (L-\epsilon)$, or $|a_n| > (L-\epsilon)^n$. Choose $\epsilon > 0$ small enough that $(L-\epsilon)|z| > 1$. Then $\sum((L-\epsilon)|z|)^n$ is a diverging geometric series and since $(L-\epsilon)|z| < |a_n||z|^n$ for infinitely many n, we have that:

$$\sum_{n=1}^{\infty} |a_n||z|^n$$

diverges. □

Remark 1.3. If it happens that $|z| = R$, then the series may or may not converge. For example, consider the power series:

(1) $\sum_{n=1}^{\infty} z^n$
(2) $\sum_{n=1}^{\infty} \frac{z^n}{n}$
(3) $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$

For (1), we have that $\frac{1}{R} = \limsup |a_n|^{\frac{1}{n}} = 1$ so $R = 1$. Then it is clear that if $|z| = R$, the series would just fail the diverging series test, i.e. $\lim_{n\to\infty} |z|^n \neq 0$. To calculate R in (2), note that:

$$\begin{aligned}
\lim_{n\to\infty} (\frac{1}{n})^{\frac{1}{n}} &= \lim_{n\to\infty} \exp(\log(\frac{1}{n})^{\frac{1}{n}}) \\
&= \lim_{n\to\infty} \exp(\frac{1}{n}\log(\frac{1}{n})) \\
&= \exp \lim_{n\to\infty} \frac{\log(\frac{1}{n})}{n} \\
&= \exp \lim_{n\to\infty} \frac{\frac{1}{n}}{1} \\
&= \exp(0) \\
&= 1
\end{aligned}$$

so $R = 1$ again. Then if $z = 1$, we just have the series $\sum_{n=1}^{\infty} \frac{z^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which does not converge. Now what if $z = -1$? Then the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the alternating series test. We can repeat the same process for (3), but this time we get that it converges since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Example 1.4. Examples of power series that converge in the whole complex plane are given by the standard **trigonometric functions**. These are defined by:

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

and

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

Theorem 1.5. *The power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defines a holomorphic function in its disc of convergence. The derivative of f is given by $f'(z) = \sum_{n=0}^{\infty} n a_{n-1} z^{n-1}$. Moreover, f' has the same radius of convergence as f.*

Proof. The assertion about the radius of convergence of f' follows from Hadamard's formula. Indeed, $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$ and therefore

$$\limsup |a_n|^{\frac{1}{n}} = \limsup |na_n|^{\frac{1}{n}},$$

so that $\sum a_n z_n$ and $\sum na_n z_n$ have the same radius of convergence, and hence so do $\sum a_n z^n$ and $\sum na_n z^{n-1}$.
To prove the first assertion, we must show that the series

$$g(z) = \sum_{n=0}^{\infty} na_n z^{n-1}$$

gives the derivative of f. For that, let R denote the radius of convergence of f, and suppose $|z_0| < r < R$. Write,

$$f(z) = S_N(z) + E_N(z)$$

where

$$S_N(z) = \sum_{n=0}^{N} a_n z^n$$

and

$$E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n$$

Then, if h is chosen so that $|z_0 + h| < r$ we have

$$\frac{f(z_0+h) - f(z)}{h} - g(z_0) = \left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N^{'}(z_0)\right)$$
$$+ (S_N^{'}(z_0) - g(z_0)) + \left(\frac{E_N(z_0+h) - E_N(z_0)}{h}\right)$$

Since $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + ab^{n-2} + b^{n-1})$, we have that

$$\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1},$$

where we have used the fact that $|z_0| < r$ and $|z_0 + h| < r$. The expression on the right is the tail end of a convergent series, since g converges absolutely on $|z| < R$. Therefore, given $\epsilon > 0$ we can find $N_1 > 0$ so that $N > N_1$ implies

$$\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| < \epsilon$$

Also, since $\lim_{N\to\infty} S_N^{'}(z_0) = g(z_0)$, we can find N_2 so that $N > N_2$ implies

$$|S_N^{'}(z_0) - g(z_0)| < \epsilon$$

If we can fix N so that $N > N_1$ and $N > N_2$ hold, then we can find $\delta > 0$ so that $|h| < \delta$ implies

$$\left|\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N^{'}(z_0)\right| < \epsilon$$

simply because the derivative of a polynomial is obtained by differentiating it term by term. Therefore,

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - g(z_0) \right| < 3\epsilon$$

whenever $|h| < \delta$. □

Since by applying this theorem on a power series yields yet another power series, we can keep on repeating this process as much as we want, even till infinity.

Theorem 1.6. *A power series is infinitely complex differentiable in its disc of convergence, and the higher derivatives are obtained by termwise differentiation.*