Totally Bounded, Compact and Complete Metric Spaces

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- We might be tempted to say this is bounded but we have no machinery to do so.
- Enter Metric Spaces
- Not all metric spaces are bounded but it would be interesting some conditions that guarantee the bounded property
- Intuitively, we want the existence of a *real* number r such that the distance between any two points in the space is less than r.

What do we mean by a "Bounded Space"?

Definition

A metric space (X, d) is bounded if there exists some number $r \in \mathbb{R}$ such that $d(x, y) \leq r$ for all $x, y \in X$. We might say r is *upper bound*, and the smallest of all such r's is said to be the *diameter* of the set X (in the context of d).

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- Subspace of a bounded space?
- Product of a bounded space?
- Compactness?

When is a metric space bounded?

Theorem

If $f : X \to Y$ is a homeomorphism and X is bounded, Y is not necessarily bounded. Subspace of a bounded metric is bounded. (At most) countable products of bounded metric spaces are bounded.

Proof:

- $(-1, 1) \cong \mathbb{R}$

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Proof:

- $(-1, 1) \cong \mathbb{R}$
- Recall the subspace is given the submetric. So if Y is a subspace of X which is a bounded space, then $d_Y(x, y) = d_X(x, y) < r$.

The Product Metric

Definition (The product metric)

Let $(X_1, d_1), \ldots, (X_n, d_n), \ldots$ be a sequence of metric spaces. For a finite product, we have $d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) =$

$$\sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2 + \ldots + d_n(x_n, y_n)^2}$$

where d is a metric on $X_1 \times \ldots \times X_n$. For an infinite product on $\Pi_{k=1}^{\infty} X_k$, we have:

$$d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(x_k, y_k)}{1 + d_k(x_k, y_k)}$$

Proof (continued)

Main Idea:

$$d(x,y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(x_k,y_k)}{1+d_k(x_k,y_k)} < \sum_{i=1}^{\infty} \frac{1}{2^k} \frac{r_k}{1+r_k} \in \mathbb{R}$$

or

$$d(x,y) = \Big(\sum_{k=1}^{n} d_k(x_k,y_k)^2\Big)^{\frac{1}{2}} < \Big(\sum_{k=1}^{n} r_k^2\Big)^{\frac{1}{2}} \in \mathbb{R}$$

No generalization possible for uncountable products of metric spaces

Compactness

We have already seen that every compact metric space is bounded. Let us recall the definition of compactness in a topological space.

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- This is the motivation for our next definition

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- We can do this for all $\epsilon > 0$
- Therefore, for all $\epsilon > 0$, we can find a finite collection of open balls at most diameter ϵ which cover X.
- Call X totally bounded

Totally Bounded

Definition

A metric space (X, d) is totally bounded if and only if for every real number $\epsilon > 0$, there exists a finite collection of open balls in X of radius ϵ whose union contains X.

Now, we check if *at least* total bounded $\implies$ bounded. If this is not true, we have a big problem with out definition :(

Relationship between bounded and totally bounded

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- Show $d(x_i, x)$ is bounded and $d(x_j, y)$ is bounded.
- Use this to show $d(y, x)$ is bounded (triangle inequality)

Proof continued

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Now for a counterexample to show bounded $\implies$ totally bounded is not true!

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- Can we cover this with a collection of open balls with diameter $\epsilon \leq 1$?

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- Think simple! What is simple? The discrete metric
- Let $|X| = \infty$. Equip this with the discrete metric d. This is bounded (why?)
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- Are we done the proof? No. Recall, we want to show the converse is not necessairly true!
- Is $(-1, 1)$ totally bounded? compact?

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- Is there some condition we can add to totally bounded to give us compactness? Hint: What went wrong in $(-1, 1)$?

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- Does it make more sense to want closedness or completeness ?

Complete Metric Spaces

Definition

A sequence $\{x_n\}$ in a metric space (X, d) is called a Cauchy sequence iff for every $\epsilon > 0$ there exists an integer N such that $d(x_m, x_n) < \epsilon$ whenever $m, n > N$.

Why Cauchy convergence and not the normal notion?

Example

Let $x_n := \frac{1}{n}$. The set (x_n) has no limit point but we can still talk about its Cauchy convergence with no reference to what it might converge to.

Complete Metric Spaces (continued)

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- Let's return to $(-1, 1)$. This is not complete, but we can see that $[-1, 1]$ is. How did I get to $[-1, 1]$?
- Every compact metric space is complete (converse not true in general)
- Is this a topological property? (There $\exists$ a one line "proof" to this)

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- How might we choose a subspace to show that X is compact?
- Don't forget the $\blacksquare$

Proof.

Let X be a totally bounded and complete metric space where d is our metric.

$\square$

Totally Bounded + Complete $\implies$ Compact

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- Pick an open cover $\{\theta_\alpha\}$ and assume it has no finite subcover.

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Let X be a totally bounded and complete metric space where d is our metric.

- Pick an open cover $\{\theta_\alpha\}$ and assume it has no finite subcover.
- Use the totally bounded condition and pick a collection of open balls of radius 1. One such ball is not covered by a finite subcollection of $\{\theta_\alpha\}$. Pick such a ball $B_d(x_0, 1)$.

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- Repeat process so we get a sequence $x_n := x_i \in B(x_i, 2^{-n})$. Note each $B(x_i, 2^{-n})$ cannot be covered by a finite subcollection of $\{\theta_\alpha\}$.

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- Repeat process so we get a sequence $x_n := x_i \in B(x_i, 2^{-n})$. Note each $B(x_i, 2^{-n})$ cannot be covered by a finite subcollection of $\{\theta_\alpha\}$.
- $x \in \theta_\alpha \implies \exists r > 0$ such that $B(x, r) \subseteq \theta_\alpha$. Show this covers infinite points of x_n which is a contradiction.

□

Compact $\implies$ Complete + Totally Bounded

We have shown Compact $\implies$ Totally Bounded.
We have shown Compact $\implies$ Complete.■

A compact meme