P1. Let $\sigma = \{a, b, c\}$ be the alphabet for this language. Then, we define the language L_1 over the alphabet σ:

$$L_1 = \{x : x \text{ does not contain the substring } bc\}$$

We claim this is a regular language generated by: $R_1 = (b^* + c^*)(ab^* + ac^* + b^*)^*$.

First, we show that $\mathcal{L}(R_1) \subseteq L_1$. Notice that there is no substring bc in $\mathcal{L}(b^* + c^*)$ and similarly for $\mathcal{L}(ab^* + ac^* + b^*)$. Furthermore, since every string of $\mathcal{L}(b^* + c^*)$ ends with either b or c and every string of $\mathcal{L}(ab^* + ac^* + b^*)$ starts with either a or b, the language $\mathcal{L}((b^* + c^*)(ab^* + ac^* + b^*))$ does not contain any strings which has a substring bc. Thus, the language $\mathcal{L}((b^* + c^*)(ab^* + ac^* + b^*)^*)$ also does not contain any strings which has a substring bc. Therefore, $\mathcal{L}(R_1) \subseteq L_1$.

Now, we show that $L_1 \subseteq \mathcal{L}(R_1)$. To do this, we show that all possible strings of size n in L_1 are in $\mathcal{L}(R_1)$ where n is any natural number (inc. 0).

P2. Let $\mathbf{MAX}(L) = \{x \in L : \text{ for any } y \in \Sigma^*,\ y \neq \epsilon, \text{ then } xy \notin L\}$. This is an operation on a language L. We show that the class of regular languages is closed under this operation and we go about doing this by supposing some DFSA M exists and accepts L, then we construct another FSA M' that accepts $\mathbf{MAX}(L)$.
Suppose a DFSA $M = (Q, \Sigma, \delta, s, F)$ exists and accepts L. Then, let F' be the set of accepting states in M such that we *cannot* reach any other accepting state of M by following some sequence of transitions i.e.

$$F' = \{x \in F : \text{ for all } y \in \Sigma^*, \text{ if } y \neq \epsilon, \text{ then } \delta^*(x, y) \notin F\}$$

Then, consider the DFSA $M' = (Q, \Sigma, \delta, s, F')$, then we know the following is true:

$$\begin{aligned} x \in \mathcal{L}(M') &= \delta^*(s, x) \in F' \\ &= \delta^*(s, x) \in F, \text{ and for all } y \in \Sigma^*, \text{ if } y \neq \epsilon, \text{ then } \delta^*(\delta^*(s, x), y) \notin F \end{aligned}$$

Then, we can simplify $\delta^*(\delta^*(s, x), y)$ to $\delta^*(s, xy)$ so that $x \in \mathcal{L}(M')$ if and only if $x \in L$, and for all $y \in \Sigma^*$, if $y \neq \epsilon$, then $xy \notin L$. However, this is simply the definition of our operation so, we get that $x \in \mathcal{L}(M')$ if and only if $x \in \mathbf{MAX}(L)$.